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## FOREWORD

This is the third issue of Notes on System Theory. The purpose of these notes is twofold. First, to provide an auxiliary publication medium for short contributions by the students and faculty who are engaged in research in systems and related areas. Second, to contribute to the development of system theory as a basic scientific discipline.

## ACKNOWLEDGMENT

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## A PITFALL IN ADAPTIVE SYSTEM DESIGN

A.R. Bergen and W.F. Colescott

Design procedures for adaptive systems often proceed along the following general lines: first, a design is made based on the assumption that the structure and all parameters of the system under study are known; second, given observations of the system as it evolves in time, a method is devised for estimating those parameters which are in fact not known initially; and third, these estimates are then used in the original design in place of the unknown parameters.

This would seem to be a reasonable approach, and in some cases has been used successfully. However, if one proceeds blindly the results can be very misleading. The following example, selected for its simplicity, demonstrates the deceptive nature of such a procedure, which, in a more complicated example, would not be as obviously transparent.

-----  
Consider a signal  $\{x_n\}$  generated by the discrete (scalar) system

$$x_{n+1} = a_n \cdot x_n \quad n = 0, 1, 2, 3, \dots \quad (1)$$

where  $\{a_n\}$  and  $x_0$  are unknown parameters.

The signal is contaminated by noise and we observe  $\{y_n\}$  where

$$y_n = x_n + u_n \quad n = 0, 1, 2, 3, \dots \quad (2)$$

The  $\{u_n\}$  are independent, gaussian, random variables with zero means. The problem is to obtain a "good" estimate  $x_j^*$  of  $x_j$  given the observed values  $y_0, y_1, y_2, \dots, y_n$ . Following the procedure outlined above:

First: Assume that the actual  $\{a_i\}$  are known; the minimum variance unbiased estimate  $x_j^*$  of  $x_j$  given  $y_0, y_1, \dots, y_n$  is

$$x_j^*(\underline{\beta}) = \frac{\beta_j}{\sum_{i=0}^n \beta_i^2} \sum_{i=0}^n \beta_i y_i \quad (3)$$

where

$$\beta_i = \prod_{k=0}^{i-1} a_k$$

$$\underline{\beta} = (\beta_0, \beta_1, \dots, \beta_n).$$

Second: Estimate the unknown  $a_i$  (or equivalently the  $\beta_i$ ) by finding the  $\beta_i^*(a_i^*)$  which minimize

$$\sum_{j=0}^n \left[ y_j - x_j^*(\underline{\beta}^*) \right]^2. \quad (4)$$

Third: It might now be argued (incorrectly) that as  $n$  increases the estimates of the  $\beta_i(a_i)$  will improve, since there is more data upon which to base the estimates. Thus, when these estimates are substituted for the  $\beta_i$  in (3), an estimate of  $x_j$  which improves as  $n$  increases is obtained.

However, if the second and third steps for this example are carried out, the following result is obtained

$$\beta_j^* = \frac{y_j}{y_0} \quad \text{and} \quad x_j^*(\underline{\beta}^*) = y_j. \quad (5)$$

The estimate for  $x_j$  is therefore just the observed  $y_j$  and the complex estimating machinery used above is not needed. Of course, due to the simplicity of the example, this result could have been seen from the start.  $y_j$  is clearly the minimum variance unbiased estimate of  $x_j$ , since with no restriction on the underlying process which generates  $\{x_n\}$ , knowledge of any  $y_j$  (or even of  $x_j$ ) can in no way help in estimating any other  $x_i$  ( $i \neq j$ ).



# CALCULATION OF CHANNEL CAPACITY

E. Eisenberg

The problem of computing the capacity of a discrete, constant channel is the following: given an  $m \times n$  matrix  $P = [p_{ij}]$ , where each row of  $P$  represents a probability distribution, to

$$\begin{aligned} &\text{find} && y = (y_1, \dots, y_m) \text{ which} \\ &\text{maximizes} && F(y) = \sum_{i,j} y_i p_{ij} \log \left[ \frac{p_{ij}}{\sum_s y_s p_{sj}} \right] \quad (1) \\ &\text{subject to} && y_i \geq 0, \quad \sum_{i=1}^m y_i = 1. \end{aligned}$$

In order to avoid inessential arguments we assume that each column of  $P$  has at least one positive entry. It should also be noted that the logarithm is taken with respect to an arbitrary fixed basis  $b > 1$ .

That the problem described by (1) is nontrivial in general can be seen from the fact that in [2, p. 136] it is stated incorrectly that  $y$  is a solution of (1) if, and only if,

$$\begin{aligned} C = \sum_{j=1}^n p_{ij} \log p_{ij} - \sum_{j=1}^n p_{ij} \log \left[ \sum_{s=1}^m y_s p_{sj} \right] \\ \text{for } i = 1, \dots, m \end{aligned} \quad (2)$$

$$\text{and } y_i \geq 0, \quad \sum_i y_i = 1.$$

That (2) need not hold for any optimizing  $y$  can be seen by taking

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix}$$

It is readily checked that in this case  $y = (1753)^{-1}(1163, 590, 0)$  is the only optimal  $y$  and the first condition in (2) is violated when  $i=3$ . Furthermore, a procedure advocated in [2, p. 139] for finding a solution of (1) which is based on (2) may yield an incorrect answer. The procedure in question is this: solve (2) without requiring  $y \geq 0$  (it is not clear that a solution will always exist); if it turns out that  $y \geq 0$ , then we have a desired  $y$ . The difficulty arises in case some  $y_i$  turn out negative. It is then suggested by Fano that one solve the  $m$  problems generated by (2) when letting one  $y_i=0$  at a time. If any of the  $y$ 's thus obtained satisfies  $y \geq 0$  then the one with largest  $F(y)$  is accepted as a solution to (1), otherwise set two of the  $y_i$ 's equal to zero, etc. It is not clear whether the first relation in (2) for that  $y_i$  which is zero, is retained or not. If it is retained one is certainly going to be in trouble as can be seen from the example cited above. If the particular relation is omitted one may get into difficulties too in case two of the  $y_i$ 's must be zero at an optimum. This will be the case when:

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{3}{8} & \frac{5}{8} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix}$$

With respect to (2) the correct statement is:  $y$  is a solution of (1) if, and only if,

$$y \geq 0, \quad \sum_{i=1}^m y_i = 1 \quad (3a)$$

$$\sum_{i=1}^m y_i p_{ij} > 0 \quad \text{all } j = 1, \dots, n \quad (3b)$$

$$F(y) \geq \sum_{j=1}^n p_{ij} \log p_{ij} - \sum_{j=1}^n p_{ij} \log \left[ \sum_{s=1}^m y_s p_{sj} \right], \quad \text{all } i. \quad (3c)$$

If one multiplies (3-c) by  $y_i$  and then sums over all  $i$  one obtains  $F(y) \geq F(y)$ , thus equality must hold in (3-c) whenever  $y_i > 0$ , but the crucial point is that when  $y_i = 0$  (3-c) must still hold, though it may turn out to be a strict inequality.

A fundamental tool needed to establish the equivalence of (1) and (3), as well as other pertinent results is the following well known

Lemma 1:

Let  $a_j, b_j$  ( $j=1, \dots, n$ ) be given numbers which satisfy  $a_j \geq 0$ ,  $b_j > 0$ , all  $j$ , and  $\sum_{j=1}^n a_j > 0$ . Then,

$$\sum_{j=1}^n a_j \log \left[ \frac{b_j}{\sum_k b_k} \right] \leq \sum_{j=1}^n a_j \log \left[ \frac{a_j}{\sum_k a_k} \right] \quad (4)$$

Furthermore, equality holds in (4) if, and only if, there exists a number  $\lambda$  such that  $a_j = \lambda b_j$  for all  $j=1, \dots, n$ . One of the important consequences of Lemma 1 is that if  $y$  and  $y'$  both solve (1). Then, even though  $y$  need not equal  $y'$ , we must have:

$$\sum_{i=1}^m y_i p_{ij} = \sum_{i=1}^m y_i' p_{ij}, \quad j = 1, \dots, n \quad (5)$$

We now turn our attention to the problem of finding a solution of (1), or of (2), in practice. It is known that the function  $F(y)$ , in (1), is a concave function, thus (1), its constraints being linear, is a convex programming problem. However, we wish to make use of a general convex programming method which requires that all functions in question possess gradients and clearly  $F$  is not differentiable at points where some  $y_i = 0$ . It is possible, though, to state a convex programming problem which is, in a sense, equivalent to (1) and which enjoys the property that all its functions are differentiable. The problem is:

find  $\alpha$  and  $\omega = (\omega_1, \dots, \omega_n)$

which maximize  $\alpha$ , subject to

$$\alpha \geq \sum_{j=1}^n p_{ij}(-\omega_j + \log p_{ij}), \quad i = 1, \dots, m \quad (6)$$

$$\sum_{j=1}^n e^{\omega_j} \leq 1.$$

The correspondence between (1) and (6) is: If  $y$  solves (1), then  $\alpha = F(y)$ ,  $\omega_j = \log \sum_{s=1}^m y_s p_{sj}$  solve (6). If  $\alpha, \omega$  solve (6), then

$$\text{let } x_j = e^{\omega_j} \left[ \sum_{k=1}^n e^{\omega_k} \right]^{-1} \quad \text{and any solution } y \text{ of}$$

$$y = (y_1, \dots, y_m) \geq 0$$

$$\sum_{i=1}^m y_i p_{ij} = x_j, \quad j = 1, \dots, n \quad (7)$$

will then be a solution of (1). The important point is that, providing  $\alpha, \omega$  solve (6), there will always exist a solution of (7). The proof of the above described equivalence between (1) and (7) is not difficult, it requires certain duality results for convex-homogeneous programming [1]. The interest of (6) lies, among other things, in the fact that in order to solve it one may apply the "Cutting plane method" for solution of convex programming problems which is described in [3, Chapt. 6]; this method, being quite general, only guarantees that a solution is obtained as a limit of a convergent process. Providing the convergence rate is reasonable such a method is quite satisfactory from the practical point of view. It would be of interest, both theoretical and practical, to have a method which yields a solution to (1) in a finite number of steps. We will now describe such a method, which is known to yield a solution after a finite number of steps when  $n$ , the number of output symbols, is two, it also may, with some slight modifications, do the same for any  $n$ ; this last is still an open question, and one of the purposes of this note is to draw attention to this problem.

The algorithm is as follows: take any  $x = (x_1, \dots, x_n)$  such that  $x_j > 0$  all  $j$ , and  $\sum_{j=1}^n x_j = 1$ . Next calculate

$$M = \min_i \left[ - \sum_{j=1}^n p_{ij} \log p_{ij} + \sum_{j=1}^n p_{ij} \log x_j \right], \quad (8)$$

let

$$I = \left\{ i \mid M = - \sum_{j=1}^n p_{ij} \log \left( \frac{p_{ij}}{x_j} \right) \right\}. \quad (9)$$

The next step is to solve the following linear programming problem:

$$\begin{aligned} &\text{find } y = (y_1, \dots, y_m) \\ &\text{which maximizes } \sum_{i=1}^m y_i \text{ subject to} \end{aligned} \quad (10)$$

$$y_i \geq 0, \quad \sum_{i=1}^m y_i p_{ij} \leq x_j \quad \text{all } j=1, \dots, n, \quad y_i = 0 \text{ if } i \in I.$$

Problem (8) always has a solution with  $y \neq 0$  and  $0 < \sum_{i=1}^m y_i \leq 1$ .

If  $\sum_{i=1}^m y_i = 1$ , then  $yP = x$  and we have a solution to (1). Otherwise,  $\mu = \sum_{i=1}^m y_i$  is strictly between 0 and 1, and  $\sum_{i=1}^m y_i p_{ij} < x_j$  for some  $j$ . Let

$$J = \left\{ j \mid \sum_{i=1}^m y_i p_{ij} = x_j \right\}$$

$$\theta = \sum_{j \in J} x_j, \quad \sigma_i = \sum_{j \in J} p_{ij}$$

$$\lambda_1 = \max_{j \notin J} \frac{1}{x_j} \sum_{i=1}^m y_i p_{ij}$$

$$\lambda_2 = \max_{\substack{i \in I, s \notin I \\ \sigma_s < \sigma_i}} \left[ \exp \frac{M-d_s}{\sigma_i - \sigma_s} \right]$$

$$\lambda = \max (\lambda_1, \lambda_2)$$

where

$$d_i = - \sum_{j=1}^n p_{ij} \log \left( \frac{p_{ij}}{x_j} \right).$$

We now modify  $x$  according to the formula,

$$x'_j = \begin{cases} x_j [\lambda + (1-\lambda)\theta]^{-1} & \text{if } j \in J \\ x_j \lambda [\lambda + (1-\lambda)\theta]^{-1} & \text{if } j \notin J \end{cases}$$

Next one recalculates (8) with the new  $x'$ , then proceeds to (9), (10), etc. This process will yield the answer (i. e., a maximum of 1 in (10)) in a finite number of steps providing  $n=2$ .

#### References

1. Eisenberg, E., "Duality in Homogeneous Programming," Proc. Am. Math. Soc., Vol. 12, No. 5 (1961), pp. 783-787.
2. Fano, Robert M., Transmission of Information - A Statistical Theory of Communications, New York: M. I. T. Press and John Wiley and Sons, 1961.
3. Wolfe, Philip, Recent Developments in Nonlinear Programming, the RAND Corporation, R-401-PR, May 1962.

# CONTROLLABILITY OF SYSTEMS WITH TIME LAGS

A. Larsen

Consider a linear, time invariant system describable by a set of  $n$  linear first order difference-differential equations of the form

$$\dot{\underline{x}}(t) = \underline{A} \underline{x}(t-1) + \underline{B} \underline{u}, \quad (1)$$

$\underline{A}$  is  $n \times n$ ,

$\underline{B}$  is  $n \times r$ ,

$t \geq 1$ ,

$\underline{u}$  is an  $r$  dimensional control.

Let  $\underline{f}(t)$  denote the initial response vector representing  $\underline{x}(t)$  during the time interval  $0 \leq t \leq 1$  and assumed continuous.  $\underline{f}(t)$ ,  $0 \leq t \leq 1$ , is analogous to an initial condition  $\underline{x}(1)$  for a system without lags.

## I. Definition:

The system (1) is controllable if and only if for all initial response vectors  $\underline{f}(t)$  which are continuous  $[0,1]$  there exists a finite time  $T > 1$  and a control vector  $\underline{u}(t)$  over  $[1, T]$  such that  $\underline{x}(T) = 0$ .

## II. Sufficient Condition for Controllability of (1):

The system (1) is controllable prior to the time  $t = k^+$  if the columns of

$$[\underline{A}^{k-1} \underline{B} \quad \underline{A}^{k-2} \underline{B} \quad \underline{A} \underline{B} \quad \underline{B}] \quad (2)$$

span the  $n$  dimensional  $\underline{x}$  space. (Note that by the Caley-Hamilton Theorem the maximum number of independent columns of (2) is determined at  $k = n$ .)



Proof;

Since the statement is only a sufficiency condition we need only concern ourselves with the terms in the solution  $\underline{x}(t)$  of (1) depending on  $\underline{u}(t)$ . If for some  $T$ ,  $\underline{u}(t)$  over the interval  $1 \leq t \leq T$  can be so chosen such that the terms involving  $\underline{u}(t)$  can represent any vector in the  $\underline{x}$  space, then they can certainly represent the negative of the vector which is the sum of the remaining terms not involving  $\underline{u}(t)$ , leaving  $\underline{x}(T) = 0$ .

Equations of the type (1) can be solved recursively. Over the interval  $1 \leq t < 2$  we have from (1),

$$\underline{x}(t) = \int_1^t \underline{A} \underline{x}(\tau-1) d\tau + \int_1^t \underline{B} \underline{u}(\tau) d\tau + \underline{x}(1)$$

or

$$\underline{x}(t) = k_1(t) + \underline{B} \int_1^t \underline{u}(\tau) d\tau,$$

where  $k_1(t)$  represents all terms in the solution not involving  $\underline{u}(t)$ . Continuing

$$\underline{x}(t) = k_2(t) + \underline{A} \underline{B} \int_2^t \underline{u}^{(1)}(\tau-1) d\tau + \underline{B} \int_2^t \underline{u}(\tau) d\tau, \quad 2 \leq t < 3$$

where again  $k_2(t)$  represents terms not involving  $\underline{u}(t)$  and we define

$$\underline{u}^{(1)} \equiv \int_1^t \underline{u}(\tau-1) d\tau.$$

Finally

$$\underline{x}(t) = \underline{k}_k(t) + \sum_{j=1}^k \underline{A}^{j-1} \underline{B} \int_k^t \underline{u}^{(j-1)}(\tau-j+1) d\tau, \quad (3)$$

$$k \leq t < k+1,$$

where in general

$$\underline{u}^{(i)} = \int_i^t \underline{u}^{(i-1)}(\tau-i) d\tau.$$

The input with arbitrary coefficient vectors  $\underline{a}_1, \dots, \underline{a}_k$

$$\underline{u}(t) = \underline{a}_1 \delta(t-k) + \underline{a}_2 \delta'(t-k+1) + \dots + \underline{a}_k \delta^{k-1}(t-1) \quad (4)$$

where  $\delta$  is the unit impulse,  $\delta'$  the doublet, etc., allows the coefficients of the columns of (2) to be chosen arbitrarily. With the input (4) the right-hand term in (3) becomes

$$\underline{A}^{k-1} \underline{B} \underline{a}_k + \underline{A}^{k-2} \underline{B} \underline{a}_{k-1} + \dots + \underline{B} \underline{a}_1.$$

Thus, if the columns of (2) span the  $\underline{x}$  space the right-hand expression in (3) can represent any vector in this space through a suitable choice of coefficients of the control (4).

#### Counterexample to Show the Condition II is Not Necessary:

Consider the system (5):

$$\dot{\underline{x}}(t) = -c \underline{x}(t-1); \quad c > 1; \quad \underline{x}(t) = \underline{f}(t), \quad 0 \leq t \leq 1. \quad (5)$$

The following can be shown:

(i) All the roots of the characteristic equation of (5) are complex with nonzero imaginary parts.

Proof:

The expression which gives the roots of the characteristic equation is

$$s + ce^{-s} = 0. \quad (6)$$

Let  $s = \sigma + j\omega$ . Then (6) becomes

$$\sigma + ce^{-\sigma} \cos \omega = 0 \quad (7a)$$

$$\omega - ce^{-\sigma} \sin \omega = 0. \quad (7b)$$

If  $\omega = 0$ , then (7a)  $\implies \sigma = -ce^{-\sigma}$  which can never be satisfied since  $c > 1$ .

(ii) Since the coefficients of (5) are all real, all roots of the characteristic equation occur in complex pairs.

It thus follows from (i) and (ii) that all terms in the solution of (5) are of the form  $p(t)e^{\sigma t} \cos(\omega t + \phi)$  where  $p(t)$  is a polynomial in  $t$  depending on the order of the root. ( $p(t)$  is of finite order since the number of roots of an analytic function with real parts on any finite interval of the real axis in the complex  $s$  plane is finite. See reference 1.)

(iii) If  $\omega_1$  is the imaginary part of a root with real part  $\sigma_1$ , then any other root with real part  $\sigma_1$  must have an imaginary part  $\pm\omega_1$ .

Proof:

Fix  $\sigma_1$ . If equation (7a) is to be satisfied for some other imaginary part  $\omega_2$ , then  $\omega_2$  must satisfy  $\omega_2 = \pm\omega_1 + 2\pi n$ ;  $n = \pm 1, \pm 2, \dots$ . However, this satisfies (7b) only if  $\omega_2 = \pm\omega_1$ .

Let  $\sigma_0$  and  $\pm\omega_0$  denote the real and imaginary parts of the roots with the largest real part having a corresponding nonvanishing

time expression in the solution of (5). Further, let  $d_p t^p e^{\sigma t} \cos(\omega_0 t + \phi_0)$  be the term in this expression with the largest integer  $p$  such that the constant  $d_p \neq 0$ . Then as  $t \rightarrow \infty$ , the amplitude of oscillation of the solution of (5) is given by  $|d_p| t^p e^{\sigma t}$ . Hence, during each half cycle, for sufficiently large finite  $t$ ,  $x(t)$  crosses the  $t$  axis. Thus,  $x(t)$  reaches zero of its own accord and the system is controllable. However, II is not satisfied.

Remarks:

The sufficiency condition II is very similar to the necessary and sufficient condition for the controllability of the analogous system without delays,

$$\dot{\underline{x}} = \underline{A} \underline{x}(t) + \underline{B} u(t). \quad (8)$$

A necessary and sufficient condition for the controllability of (8) is that the columns of (2) span the  $\underline{x}$  space with  $k=n$ . The difference is that with system (1) one must in general wait until time  $t = n$  before (2) fills out to its maximum rank, whereas in the case of system (8) the maximum rank is attained immediately.

The other major difference between (1) and (8) regarding controllability is the nonnecessity of II if the system has lags. This is due to the possibility of an oscillation in a simple first-order difference-differential equation as the counterexample shows.

Reference:

1. Caratheodory, C., Theory of Functions, New York: Chelsea Publishing Co., 1954 (Translation).

## MACHINES THAT GENERATE REGULAR MARKOV CHAINS

D. Moorehead

Any finite state machine will, by virtue of the Markov character of the "next state" function, generate a finite Markov chain when a distribution is defined over the input alphabet. Of particular interest is that class of machines which will generate a "regular" Markov chain.<sup>+</sup> Such machines have the property that the probability distribution vector defined over the states converges to a vector constant in time. Given a Moore machine model we may identify an output symbol with a subset of the state space. This in turn permits us to specify a distribution vector over the output that is constant in time.

Suppose, then, a Moore machine  $M$  with the following characterization: an input alphabet set

$$U = \{a_1, a_2, \dots, a_r\},$$

an output alphabet set

$$Y = \{\beta_1, \beta_2, \dots, \beta_s\},$$

and a state space

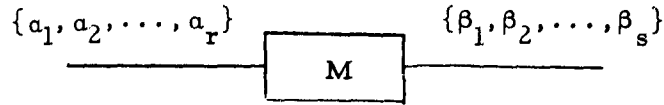
$$S = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$$

$$u_t = \text{input at time } t$$

$$y_t = \text{output at time } t.$$

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<sup>+</sup>A. Gill, "Synthesis of Probability Transformers," Journal of the Franklin Institute (July 1962).



There is a probability distribution vector  $\pi_U$  defined over the input alphabet,

$$\pi_U = \{\pi_1, \pi_2, \dots, \pi_r\}; \quad \pi_i = P\{u_t = a_i\} > 0.$$

The sequence of inputs are independent and identically distributed.

For any unordered pair of states  $(\sigma_i, \sigma_j)$ ,  $\sigma_i, \sigma_j \in S$ , there exists a subset of  $U$  (possibly empty) say  $U_{ij}$ , such that  $\sigma_i$  will be mapped into  $\sigma_j$  under  $U_{ij}$ . We denote this by

$$\sigma_i \xrightarrow{U_{ij}} \sigma_j.$$

Let  $s_t$  be the state of machine  $S$  at time  $t$ . Then

$$\begin{aligned} P_{ij} &= P\{s_{t+1} = \sigma_j \mid s_t = \sigma_i\} = P\{u_t \in U_{ij}\} \\ &= \sum_{a_i \in U_{ij}} \pi_i. \end{aligned}$$

This allows us to construct a stochastic matrix,

$$P = (P_{ij}).$$

From Markov chain theory we have that  $P$  describes a regular Markov chain if and only if  $\exists n$  such that  $P^n$  is a positive matrix. Then,  $\lim_{n \rightarrow \infty} P^n$  exists. We call this limit  $T$ . Further, it has the property that all its row vectors are the same vector  $F$ :

$$T = \begin{pmatrix} F \\ F \\ \vdots \\ F \end{pmatrix}, \quad \sum F_i = 1, \quad F_i > 0$$

$$F_i = P\{s_{t=\infty} = \sigma_i\}.$$

The vector  $F$  is an eigenvector of  $P$  with eigenvalue 1:

$$FP = F.$$

These relations have the implication that as time approaches infinity the probability that the system is in a given state is nonzero and independent of the initial state.

With each state is associated some output symbol  $\beta_i$ . Thus for each  $\beta_i$  there exists a subset of states  $S_{\beta_i}$  such that

$$P\{y_t = \beta_i \mid s_t \in S_{\beta_i}\} = 1.$$

Therefore,

$$P\{y_t = \beta_i\} = \sum_{\sigma_i \in S_{\beta_i}} P\{s_t = \sigma_i\}$$

and

$$P\{y_{t=\infty} = \beta_i\} = \sum_{\sigma_j \in S_{\beta_i}} F_j.$$

We may speak of an  $s$  component vector  $q$  over the output where

$$q_i = P\{y_{t=\infty} = \beta_i\}.$$

From convergence considerations,  $\forall \epsilon, \epsilon > 0, \exists n$  such that,

$$\left| P\{y_t = \beta_i\} - P\{y_{t=\infty} = \beta_i\} \right| < \epsilon;$$

i. e., if we observe the output of the machine of the aforementioned type at intervals of time sufficiently long we can achieve an output alphabet distribution vector as near constant as we please.

We make the following definition: two machines  $A$  and  $B$  are equivalent in the regular Markov sense iff for all distributions over the input alphabet the probability of the output, being any member of the output alphabet, is the same for  $A$  and  $B$ . Let  $\pi_U$  be the

distribution over the input alphabet. Regular Markov equivalence will be denoted by  $\{A \xrightarrow{\text{R.M.}} B\}$ . Then the above definition becomes

$$\{A \xrightarrow{\text{R.M.}} B\} \iff \left\{ \forall \pi_U, \forall \beta_i \in Y, P\{y_t^A = \beta_i\} = P\{y_t^B = \beta_i\} \right\}$$

where  $y_t^A$  and  $y_t^B$  are the outputs from A and B, respectively, at time t.

Theorem:

If two deterministic machines in the regular Markov class are equivalent, then they are R.M. equivalent.

Proof:

Let A and B be two machines in the regular Markov class that are equivalent. Assume that the initial states of A and B are equivalent. Let the same input be fed to both machines. Then the outputs will be identical. Since the outputs are identical, the distributions over the outputs are the same and, hence, must tend to the same limit. Since the limiting distributions are independent of the initial state in a regular Markov chain, the limiting distribution attained in this way is the limiting distribution for the process. Q. E. D.

The following examples of two machines, A and B, illustrate that the converse of this theorem does not hold, i. e., machines can be R.M. equivalent, but not equivalent in the Moore sense.

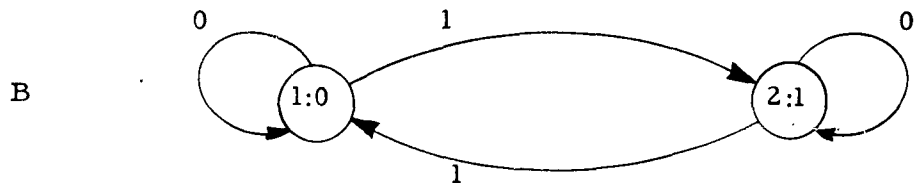
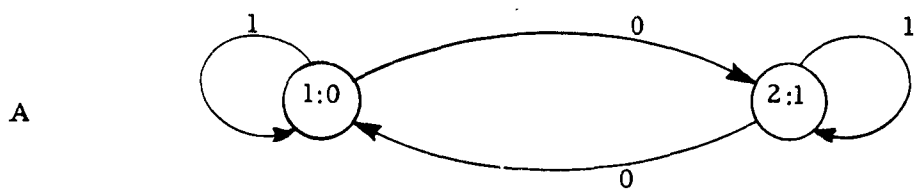
Example:

$$U = \{0, 1\}$$

$$Y = \{0, 1\}$$

$$S = \{1, 2\}.$$





Let  $P\{u_t = 1, \forall t\} = P,$

$P\{u_t = 0, \forall t\} = 1-P$

$$P_A = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} P & 1-P \\ 1-P & P \end{pmatrix} \end{matrix}$$

$$P_B = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} 1-P & P \\ P & 1-P \end{pmatrix} \end{matrix}$$

$P_A$  and  $P_B$  are the stochastic matrices for A and B, respectively.  
Let  $F^A$  and  $F^B$  be their respective fixed vectors: i. e.,

$$F^A P_A = F^A, \quad F^B P_B = F^B$$

$$F_1^A + F_2^A = 1, \quad F_1^B + F_2^B = 1.$$

A solution to the above set of equations yields

$$F_1^A = F_2^A = F_1^B = F_2^B = \frac{1}{2}.$$

Thus, A and B are R.M. equivalent but Moore distinguishable.

A word is in order regarding the conditions on a machine such that it falls into the class of regular Markov machines. A transliteration of the regular Markov property into machine structure provides us with the following:

Definition:

A machine may generate a regular Markov chain iff there exists an integer  $k$  such that any state is reachable from any other state in  $k$  transitions.

It is obvious that the above definition implies strong connectedness. This allows us when dealing with machines of the regular Markov type to speak of equivalence under a simple experiment.

A less stringent and possibly more useful definition of equivalence is one demanding equivalence only under a particular input distribution. We propose the following.

Definition:

Two machines are R.M. equivalent under  $\pi_U$  iff  $\forall i$

$$P\{y_{t=\infty}^A = \beta_i\} = P\{y_{t=\infty}^B = \beta_i\}.$$

Denote this equivalence by  $\{A \xrightarrow[\pi_U]{\text{R.M.}} B\}$ . Then,

$$\{A \xrightarrow[\pi_U]{\text{R.M.}} B\} \iff \left\{ \forall i \ P\{y_{t=\infty}^A = \beta_i\} = P\{y_{t=\infty}^B = \beta_i\} \right\}.$$

A simple test for equivalence here is just to solve the homogeneous equations for each machine,

$$F^A P_A = F^A, \quad F^B P_B = F^B,$$

and compare output probabilities. This definition permits one to draw from a much larger class of machines and provides a very simple test for equivalence.

Conditions on a Machine such that it Generates a Transition Matrix over the Output:

Consider the partition generated by the requirement that each state generate the same output. The class defined for each  $\beta_i$  becomes,

$$S_{\beta_i} = \{\sigma_i : \beta_i = F(\sigma_i)\}.$$

If we can say further that

$$P\{s_{t+1} \in S_{\beta_i} \mid s_t = \sigma_p \in S_{\beta_j}\} = P\{s_{t+1} \in S_{\beta_i} \mid s_t = \sigma_q \in S_{\beta_j}\} \forall \sigma_p, \sigma_q \in S_{\beta_j}$$

we may speak of the stationary transition probability from set  $S_j$  to the set  $S_i$ ,

$$P\{s_{t+1} \in S_{\beta_i} \mid s_t \in S_{\beta_j}\} = P\{s_{t+1} \in S_{\beta_i} \mid s_t = \sigma_p \in S_{\beta_j}\} \forall \sigma_p \in S_{\beta_j}.$$

The transition matrix  $P_Y$  on the output is

$$P_Y = \begin{matrix} & \beta_1 & \beta_2 & \dots & \beta_s \\ \begin{matrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_s \end{matrix} & \left( \begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \end{array} \right) & = (P\{s_{t+1} \in S_{\beta_i} \mid s_t \in S_{\beta_j}\}) \\ & & & & \end{matrix}$$

$$= \sum_{j \in S_{\beta_j}} P_{ij}, \forall i \in S_{\beta_i}.$$

This establishes a machine  $(M', \text{ say})$  with  $s$  states. For  $\sigma_i, \sigma_j \in M'$ , we make the correspondences

$$S_{\beta_i} \longrightarrow \sigma_i, \quad S_{\beta_j} \longrightarrow \sigma_j.$$

Select  $U_{ij} = \{a_1, \dots, a_l\}$   $l \leq r$  subject to the condition,

$$\sum_{a_i \in U_{ij}} P\{u_t = a_i\} = P\{s_{t+1} \in S_{\beta_i} \mid s_t \in S_{\beta_j}\}.$$

For  $\forall \sigma_j \in S_{\beta_j}, \forall \sigma_i \in S_{\beta_i}, U_{ij}$  is all  $a_i$  such that

$$\sigma_j = g(\sigma_i, a_i).$$

#### Regular Markov Mealy Machines:

It is clear that the state structure requirement on the Mealy machine for regular Markov chain generation is the same as that for the Moore machine. The procedure for computing the output alphabet probabilities differs somewhat. Consider the set of all those states  $S_{\beta_i}$  which could, in conjunction with an appropriate input, give rise to  $\beta_i$ . Let the set of inputs that produces  $\beta_i$  for given state  $\sigma_k$  be designated  $\pi_{k\beta_i}$ . Then

$$P\{y_{t=\infty} = \beta_i\} = \sum_{\substack{\sigma_k \in S_{\beta_i} \\ \pi_{kj} \in \pi_{k\beta_i}}} F_k \pi_{kj}$$

Since there exists an equivalent Moore machine for any Mealy machine,<sup>+</sup> all evaluations for outputs may be carried out for the Moore machine equivalent. The above theorem assures that the output probabilities will be equal. Hence, the Mealy machine case does not pose a distinctly different problem.

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<sup>+</sup> A. Gill, "Comparison of Finite-State Models," IRE Trans. on Circuit Theory, CT-7 (1960), pp. 178-179.

# ANOTHER LOOK AT THE SAMPLING THEOREM AND ITS EXTENSIONS<sup>+</sup>

J. F. A. Ormsby

## Introduction:

The sampling principle had its origins in the analytic interpolation function theory of Euler and Lagrange. Later in this century further extensions and refinements were carried out in general mathematical treatments by a number of investigators.<sup>++</sup> The connection of these treatments with the sampling principle in signal theory as stated by Shannon [1], [2], is the cardinal series given by

$$f(x) = \sum_{n=-\infty}^{\infty} y_n \frac{\sin \pi(x-n)}{\pi(x-n)} \quad (1)$$

The properties of this series in general interpolation theory are discussed for example by J. M. and E. T. Whittaker, W. L. Ferrar, and I. J. Schoenberg.<sup>++</sup>

It is interesting to note that Cauchy [3] as early as 1841, verbally suggested the possibility of an extension of (1) to essentially nonuniform sampling. In this connection Yen [4] has recently detailed this possibility and generalized its statement to a group of sampling theorems for nonuniform sampling (in time).

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<sup>+</sup>This note is a somewhat condensed version of material originally presented as a term paper for the course Elect. Engr. 298 given by Prof. C. A. DeSoer at U. C. in the Fall, 1960, semester. A few later references have been added.

<sup>++</sup>These include Ch. J. de la Vallée Poussin, E. T. and J. M. Whittaker, W. A. Jenkins, T. N. E. Grenville, I. J. Schoenberg, W. L. Ferrar, and N. E. Nörlund. The book by J. F. Steffensen, Interpolation, New York: Chelsea, 1950, summarizes some of this work.

Although earlier suggestions of the implications of the sampling principle in communication theory were made by Nyquist and Gabor, its emergence as a widely used and accepted concept awaited Shannon. Then its prominence coincided in time with the recognition of the stochastic estimation concepts as advanced by Wiener. These two areas find a mutual application for example in the consideration of discrete parameter (time) series. An interpretation of equation (1) in a stochastic sense was treated in a rigorous fashion recently by Balakrishnan [5] showing that pointwise limits carry over to limits in the mean.<sup>+</sup>

We are concerned here with the sampling principle and its generalizations in the nonstochastic case that is for a (single) sure signal. However, the stochastic interpretation (as defined in [5]) of such results is direct.

In reviewing recent papers on the sampling theorem and its generalizations in the information theory literature, one quickly becomes aware of the variety of approaches used for analysis. Indeed, the proofs are in large part specialized and very often cumbersome. For example, one approach frequently used sets up spectra models with spectra constraints to prevent frequency aliasing. The satisfying of the resulting conditions on the (time) samples then provides the desired generalization.

Our purpose here will be to exhibit an approach or at least suggest use of a point of view which encompasses all these generalizations and suggests additional extensions. Along with this discussion, our note will also serve as a review of recent contributions in sampling theory.

<sup>+</sup>For a very extensive treatment of sampling in the stochastic case, see S. P. Lloyd, "A Sampling Theorem for Stationary (Wide Sense) Stochastic Processes," Trans. Amer. Math. Soc., Vol. 92 (July 1959), pp. 1-12 (also as Bell System Monograph 3433).

However, before entering upon this it seems appropriate to present a short review of the basic sampling theorem to place the extensions in the proper setting.

#### Review of the Basic Sampling Theorem:

The basic sampling principle (for uniform sampling) can be restated from (1) for comparison with more general forms as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta t) \frac{\sin \pi/\Delta t (t-n\Delta t)}{\pi/\Delta t (t-n\Delta t)} \quad (2)$$

where  $\Delta t$  is the sampling (time) interval. For  $\Delta t = 1/2W$ , equation (2) gives a unique error free representation of  $f(t)$  when the spectrum of  $f(t)$  is confined to the frequency interval  $(-W, W)$ .

We now briefly motivate (2) by two formal approaches (a) and (b) and a more rigorous treatment (c). Questions of rigor in (a) and (b) enter in terms of interchanges of limits (in sums and integrals). However, since (a) is the more conventional approach and (b) has the appeal of operator manipulation, we include them.

$$(a) \quad \text{For } F(\omega) = 0, \quad |\omega| > 2\pi W, \quad F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt.$$

Expanding  $F(\omega)$  in a Fourier series,

$$\begin{aligned} F(\omega) &= \sum_n a_n e^{i\omega n/2W}; & |\omega| &\leq 2\pi W \\ &= 0; & |\omega| &> 2\pi W \end{aligned}$$

with

$$a_n = \frac{1}{4\pi W} \int_{-2\pi W}^{2\pi W} F(\omega) e^{i\omega n/2W} d\omega.$$

From

$$f(t) = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} F(\omega) e^{i\omega t},$$

we have on letting  $t = n/2W$ ,

$$a_n = \frac{1}{2W} f(n/2W).$$

Taking inverse Fourier transforms on the series expansion of  $F(\omega)$ , we get

$$f(t) = \sum_{n=-\infty}^{\infty} a_n \frac{\sin \pi(2Wt+n)}{\pi(2Wt+n)} = \sum_{n=-\infty}^{\infty} f(n/2W) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}.$$

$$(b) \quad F(f) = \hat{F}(f) [\tau \operatorname{rect}(f/2W)], \quad \tau = 1/2W, \quad f = \omega/2\pi, \quad \text{and}$$

$$\hat{F}(f) = \sum_{n=-\infty}^{\infty} \frac{1}{\tau} F(f - \frac{n}{\tau}).$$

Taking inverse Fourier transforms  $\mathcal{F}^{-1}$ , we have with  $\mathcal{F}g(t) = F(f)$ ,

$$g(t) = \mathcal{F}^{-1} \left\{ \hat{F}(f) [\tau \operatorname{rect}(f/2W)] \right\} = [\mathcal{F}^{-1}(\hat{F}(f))] * \operatorname{sinc} 2\pi Wt^{\dagger}.$$

Now

$$\hat{F}(f) = F(f) * \sum_{n=-\infty}^{\infty} \frac{1}{\tau} \delta(f - \frac{n}{\tau}),$$

so that

$$\mathcal{F}^{-1}(\hat{F}(f)) = (\mathcal{F}^{-1}F(f)) * \sum_{m=-\infty}^{\infty} \delta(t - \frac{m}{2W}).$$

<sup>†</sup>The symbol  $*$  stands for the convolution operation.



We have, therefore,

$$\begin{aligned} g(t) &= \left[ g(t) \sum_{-\infty}^{\infty} \delta\left(t - \frac{n}{2W}\right) \right] * \text{sinc } 2\pi W t \\ &= \sum_{-\infty}^{\infty} g(n/2W) \text{sinc } 2\pi W\left(t - \frac{n}{2W}\right). \end{aligned}$$

(c) This approach of expanding  $e^{i2\pi ft}$  in a Fourier series is due to Parzen. The style of proof given here is that found in Balakrishnan [5] or more recently Beutler [6].

Again using

$$g(t) = \int_{-W}^W e^{i2\pi ft} F(f) df,$$

we expand

$$e^{i2\pi ft} = \sum_{-\infty}^{\infty} a_n(t) e^{in\pi f/W}, \quad f \in (-W, W)$$

with

$$\begin{aligned} a_n(t) &= \frac{1}{2\pi} \int_{-W}^W e^{i2\pi ft} e^{-in\pi f/W} df \\ &= \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}. \end{aligned}$$

Thus,

$$g(t) = \int_{-W}^W \left( \lim_{N \rightarrow \infty} \sum_{-N}^N a_n(t) e^{in\pi f/W} \right) F(f) df.$$

Since  $e^{i2\pi ft}$  is absolutely continuous for  $f \in (-W, W)$ ,  $\sum_{-N}^N a_n(t) e^{in\pi f/W}$  converges boundedly. Denoting  $Y_N = \sum_{-N}^N$ , we have  $Y_N \xrightarrow{N} Y$  with  $Y_N$  bounded for all  $N$ .

Using the zero measure of the singleton  $\{-W\}$  and  $\{W\}$  on the  $f$  line to give  $\int_{f \in (-W, W)} = \int_{f \in [-W, W]}$  and applying the dominated convergence theorem we have

$$\begin{aligned} g(t) &= \int_{-W}^W (\lim_N Y_N) F(f) df = \lim_N \int_{-W}^W Y_N F(f) df \\ &= \lim_N \int_{-W}^W \left( \sum_{-N}^N a_n(t) e^{in\pi f/W} \right) F(f) df \\ &= \lim_N \sum_{-N}^N a_n \int_{-W}^W e^{in\pi f/W} F(f) df = \sum_{-\infty}^{\infty} g(n/2W) a_n \\ &= \sum_{-\infty}^{\infty} g(n/2W) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}. \end{aligned}$$

#### Recent Generalizations of the Sampling Theorem:

Two basic types of generalizations of (2) involve considerations on what samples are taken and when the samples are taken. We call these Types I and II. For identification purposes with respect to the approach of the next section, we will also label a Type III.

Type I: This generalization of (2) substitutes more sample values for fewer sample times by using samples not only of the function itself but also its derivatives. Using up to the  $K^{\text{th}}$  order derivative, its form can be given as

$$f(t) = \sum_{n=-\infty}^{\infty} \left( \sum_{l=0}^K \frac{(t-n\Delta t)^l}{l!} \left. \frac{d^l f(t)}{dt^l} \right|_{t=n\Delta t} \right) \left( \frac{\sin \pi/\Delta t (t-n\Delta t)}{\pi/\Delta t (t-n\Delta t)} \right)^{K+1} \quad (3)$$

where  $\Delta t = (K+1)/2W$  and  $f(t)$  is bandlimited to  $(-W, W)$  as in (2).

Shannon [1] first remarked about the possibility of such a generalization. Then Fogel [7] supplied a spectrum argument justifying the conjecture using considerations for specifying the spectrum uniquely in terms of a solution to a system of equations. The approach, in fact, involved inducing spectrum folding. Later, Fogel and Jagerman [8] gave a statement of (3) for  $K=1$  in connection with extending (2) for  $t$ , a complex variable. More recently, Abramson and Linden [9] gave (3) for arbitrary  $K$ . However, their approach and method of proof is cumbersome. They assume the result and show its validity by proving the difference between the two sides of (3) is zero. The proof rests on showing that a certain function is identically zero if it has a bandlimited spectrum and if its first  $K$  derivatives are zero at the sample points. It involves the detail of a Vandermonde matrix analysis.

Type II: This generalization involves taking sample values at nonuniform times (or at least not entirely uniform). The first particular result along this line appears to be the work of Kohlenberg [10] in connection with bandpass signal spectra. The resulting sample times have a high degree of uniformity in this case.

Although a sampling theorem, if it exists, could be written in a form reflecting the particular sampling time pattern for each case considered, a general principle can be given following Yen [4]. The principle is stated by Yen as a generalization of his

Theorems I and III. Based on this, its validity would appear restricted to the cases covered by just these theorems. The principle which indeed is an extension of the early comment of Cauchy [3] can be given as follows.

If a signal is a magnitude-time function and if time is divided into equal intervals of  $T$  seconds where  $T=N/2W$  and  $N$  instantaneous samples are taken from each interval in any manner, then a knowledge of the magnitude of each sample and the instant at which each sample is taken determines the original signal uniquely. If the number of samples is less than that stated above in any interval, the signal becomes underspecified, i. e., additional conditions must be employed before the signal is determined uniquely. On the other hand, if the sample number is more than  $N$  in any interval, the signal becomes overspecified, that is, the sample values cannot be arbitrarily assigned, but must satisfy a certain number of consistency conditions. The signal in the above is bandlimited to  $(-W, W)$ .

The approach to be suggested in the next section allows Type II generalization to follow naturally without necessarily tying it to a particular structure such as in Yen. Further the approach of Kohlenberg, like Fogel for Type I, contrived the required system of equations on the spectrum. In reference [10] a double sequence of sampling points (second order sampling) is introduced. In the next section an attempt is made to eliminate the need for these particular contrivances by presenting a unified point of view.

Type III: As noted above Fogel and Jagerman [8] considered extending the basic theorem (2) (and (3) for  $K=1$ ) to the case where  $t$  is complex. The justification applying contour integration is indeed cumbersome. Such an extension and the contour integral equivalent appear naturally in the approach to be discussed.

In closing this section, a few additional remarks can be made to fill out the picture of extensions in sampling theory. Aside from the justification of (2) in the stochastic case by Balakrishnan who incidentally considers both absolutely and nonabsolutely continuous spectra, other extensions of sampling methods can be mentioned. These include random time sampling.

If the sample time differences are not independent (statistically), then errors may result. Consider for example the case where the average sample rate is proper (to prevent nonarbitrariness or folding) but there exists a random jitter. The jitter at successive times of sampling is taken independent but the difference in sample times is not independent. Then the resulting power spectrum is attenuated by a factor which is the square of the characteristic function of the sample time jitter component distribution [11], [12].

Recently Shapiro and Silverman [13], treating the case of independent sample time differences, produced sampling without spectrum folding (aliasing) when no restrictions are placed on the sampling interval in terms of the bandwidth limitation. The sample time difference in this case followed a Poisson distribution. This result hinges on the characteristic function of the sample difference times being single valued. The close connection of this Fourier transform (characteristic function) and the Fourier transform of the quantity  $\phi_k(\lambda)$  in the extended fundamental form (see equation (7)) discussed in the next section is to be noted. Schoenberg [14] takes the Fourier transform of the Lagrange coefficient  $\phi_k(\lambda)$ , calling this the characteristic function and establishes a differentiation between smoothing (interpolation error) and exact interpolation on the basis of the range and values taken by this characteristic function.

These connections to  $\phi_k(\lambda)$  are not surprising since after all, the basic ingredients must lie in the  $\phi_k(\lambda)$  or the distribution of the  $\lambda_k$  (see next section) since the rest of the fundamental form extended (see equation (7)) involves the arbitrary function  $f(\lambda)$ .

Finally we may consider sampling theorems for multidimensional signals. For example, Kailath [15] has specified sampling formulae applied to a function of two variables with frequency restrictions on one or both of these variables. He chose for convenience to use the Hilbert transform form as used by Woodward [16]. The convenience but lack of direct interpretation of this approach can be illustrated for example by comparing the difficulty in obtaining forms of Kohlenberg [10] completely defined in time and signal values with those of Woodward based on Hilbert transforms.

In dealing with multidimensional (multivariable) sampling, one could also seek theorems reflecting the correlation between the components (variables). A general interpolation form in more than one variable as a further extension of formula (7) of the next section should then be able to encompass sampling in multidimensional cases in the natural way that (7) does for one dimensional sampling.

#### Another Approach to Sampling Theorems:

Considering the great variety and detail in approach together with the arbitrariness and cumbersome nature of the proofs as indicated in the last section for the various theorems, it could be hoped that another more general point of view might exist which would encompass the various generalizations and by its generality reduce the need for specialized detail. The following approach seems to offer a solution.

From general interpolation theory and related results<sup>+</sup> on

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<sup>+</sup>The material of interest here is part of the spectral theory of linear operators. The area of formulae of specific concern involving functions of matrices was presented as part of the seminar course of Prof. Desoer already noted. This material is also found in reference [17] especially sections 18 and 19.

interpolating polynomials and their connection with the minimum and characteristic polynomials, there emerges an approach which encompasses the basic theorem (2) and, for example, its Type I, II, and III extensions.

It will not be the intent of this note to justify the steps taken to extend the interpretation of the fundamental formulae given below together with its equivalents and needed properties in order to associate with the requirements of the sampling theory. However, the approach will be outlined in some detail and examples of its application to obtaining Type I, II, and III generalizations will be noted. In addition, its use on other generalizations and its potential to extend even further will be discussed.

Our starting point is the fundamental formulae<sup>+</sup> given by

$$f(\underline{A}) = \sum_{k=1}^s \sum_{l=0}^{m_k-1} \frac{(\underline{A} - \lambda_k \underline{I})^l}{l!} f^{(l)}(\lambda_k) \underline{E}_k \quad (4)$$

- where (i)  $\lambda$  is, in general, a complex variable  
(ii)  $\underline{A}$  is some matrix (or equivalently, a linear transformation in finite dimensional space) with specified spectrum (eigenvalues)  $\{\lambda_1, \lambda_2, \dots, \lambda_s\}$ .  
(iii)  $f$  is some function analytic on an open set containing  $\{\lambda_k\}_1^s$   
(iv)  $m_k$  is the multiplicity of  $\lambda_k$ ;  $k = 1, \dots, s$ .  
(v)  $\underline{E}_k = \phi_k(\lambda) = \frac{\psi(\lambda)}{(\lambda - \lambda_k)^{m_k}} n_k(\lambda)$  with  $n_k(\lambda)$  a polynomial

of order  $m_k - 1$  in  $\lambda$  and  $\lambda_k$  and  $\psi(\lambda)$  the minimum polynomial.  $\underline{E}_k$  has the range of the generalized null space associated with  $\lambda_k$  and in fact is the projection operator into this null space.

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<sup>+</sup>See the previous footnote.

We first take  $\underline{A} \rightarrow \lambda$  to get

$$f(\lambda) = \sum_{k=1}^s \sum_{l=0}^{m_k-1} \left( \frac{(\lambda - \lambda_k)^l f^{(l)}(\lambda_k)}{l!} \phi_k(\lambda) \right) \quad (5)$$

From the theory we may also write, after taking  $\underline{A} \rightarrow \lambda$ ,

$$f(\lambda) = \sum_{k=1}^s \sum_{l=0}^{m_k-1} ( ) = \frac{1}{2\pi i} \oint_C f(\sigma) \rho(\lambda) d\sigma \quad (6)$$

where  $C$  encloses the spectrum inside the domain of analyticity of  $f$  and  $\rho(\lambda) = \frac{1}{\sigma - \lambda}$ . Finally from the general structure after again taking  $\underline{A} \rightarrow \lambda$ ,

$$f(\lambda) = 0 \iff f^{(l)}(\lambda_k) = 0 \quad \begin{matrix} l = 0, 1, \dots, m_k-1 \\ k = 1, 2, \dots, s \end{matrix} \quad (7)$$

With these results, we now proceed to apply this theory to the sampling problem. The basic theorem (2) as well as Type I, II, and III generalizations can proceed directly on a formal basis by an extension of (6).

We next renumber the  $\lambda_k$  so that  $\frac{s-1}{2} = N$  for  $s$  odd and  $\frac{s}{2} - 1 = N$  for  $s$  even so that with  $N \rightarrow \infty$ , we write

$$f(\lambda) = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{m_k-1} \frac{(\lambda - \lambda_k)^l f^{(l)}(\lambda_k)}{l!} \phi_k(\lambda) = \frac{1}{2\pi i} \oint_{C_{\infty}} \frac{f(\sigma)}{\sigma - \lambda} d\sigma \quad (7)$$

where  $C_{\infty}$  is some contour taken as a limit of  $C_N$  as  $N \rightarrow \infty$ , with  $C_N$  enclosing the spectrum  $\{\lambda_k\}_{-N}^N$ .

To have a closer connection with the nomenclature appearing in sampling theory literature, we let  $\xi = \sigma$  and  $z = \lambda$ . In applying (7)



for the basic sampling theorem, we take  $m_k = 1$ , for all  $k$  corresponding to first order sampling where only one sample value is taken at each sample time. The association between eigenvalues and complex sample times is made. Before exhibiting the form of  $\phi_k(z)$ , we note the following properties of  $\phi_k(z)$  which make the representations unique for each spectrum  $\{z_k\}$  chosen, as a result of a requirement of the minimum degree polynomial:

$$\phi_k(z_k) = 1; \quad \phi_k^{(\ell)}(z_k) = 0; \quad \ell = 1, 2, \dots, m_{k-1}$$

$$\phi_k^{(\ell)}(z_j) = 0; \quad \ell = 1, 2, \dots, m_{j-1}$$

$$j = 1, 2, \dots, k-1, k+1, \dots, s$$

$$= -N, \dots, k-1, k+1, \dots, N$$

with  $N \longrightarrow \infty$  in our case.

With  $m_k = 1$ , we have,

$$f(z) = \sum_{-\infty}^{\infty} f(z_k) \phi_k(z).$$

Now<sup>+</sup>

$$\phi_k(z) = \frac{\prod_{j=-\infty}^{\infty} (z - z_j)}{\prod_{j=-\infty}^{\infty} (z_k - z_j)} = \frac{\prod_{j=-\infty}^{\infty} (z - z_j)}{(z - z_k) \prod_{j=-\infty}^{\infty} (z_k - z_j)}$$

$$= \frac{\sin \pi/hz}{(z - kh)(\sin \pi/hz)'} \Big|_{z=kh} \quad \text{for } z_j = jh, \quad h, \text{ in general, complex}$$

<sup>+</sup>  $\pi_j^!$  means  $j \neq k$ , in  $\phi_k(z)$ .

$$= \frac{(-1)^k \sin \pi/hz}{\pi/h (z-kh)} = \frac{\sin \pi/h (z-kh)}{\pi/h (z-kh)}.$$

Therefore,

$$f(z) = \sum_{-\infty}^{\infty} f(kh) \frac{\sin \pi/h (z-kh)}{\pi/h (z-kh)}, \quad \text{which is (2) form.}$$

In view of the later discussion concerning Type III generalization, we note that this result is immediately equal to

$$\frac{1}{2\pi i} \oint_{C_{\infty}} \frac{f(\xi)}{\xi - z} d\xi$$

where  $C_{\infty}$  encloses  $\{\lambda_k\}_{-N}^N$  as  $N \rightarrow \infty$ .

The application of (7) to Type I generalization carries over in same way. We illustrate for  $m_k = 2$  (i. e.,  $l=0,1$ ) to give repeated sample points. Then  $\phi_k(z)$  becomes the square of its value in the basic theorem. Also with  $l=0,1$  both  $f(z_k)$  and  $f^{(1)}(z_k)$  appear. We have at once using  $\phi_k(z) \rightarrow [\sin(\pi/h (z-kh))]^2$  as  $N \rightarrow \infty$ ,

$$f(z) = \sum_{-\infty}^{\infty} [f(kh) + (z-kh)f^{(1)}(kh)] \left( \frac{\sin \pi/h (z-kh)}{\pi/h (z-kh)} \right)^2$$

which is (3) form for  $K=1$ . The form for the  $K^{\text{th}}$  order derivative case as given in (3) is just as simply obtained.

As noted previously, the detailed proof of Abramson and Linden [9] involved showing that a function is identically zero if it has a bandlimited frequency spectrum and if its first  $K$  derivatives

<sup>+</sup> See footnote on previous page.

are zero at the sample points. Such a result appears direct from the general interpolation structure as given by (7). The bandlimited property is related to the analyticity of  $f$  and the spacing between successive  $z_k$ , specifying  $h$ .

We now turn to Type II generalization. Since the problem of unique determination for the various nonuniform sample point distributions can be answered in a natural way using the general inclusive form (7), we need only turn to the particular expression which this form produces for a particular case. We give as an example the case of Theorem I of Yen [4] (this case together with his Theorem III constituted the area of application given by reference [4] for Type II as noted earlier). The development in [4] is tedious and indirect lacking the natural development of the form of the sampling formula using the point of view suggested by (7).

Treating the real time case, we take  $z=t$  and consider  $N$  sample times altered from an otherwise uniform sampling with  $h = 1/2 W$ . The uniform sample times are at  $n/2W$ , while the altered times are at  $\tau_p$ ,  $p = 1, 2, \dots, N$ . The holes left by the altered times can be given as  $\tau_p/2W$  using the notation of Yen. To correspond more closely to his notation, we take  $k=m$ . Now for all the nonuniform sampling situations treated by Yen and Kohlenberg, our  $m_k=1$ .

We then have from (7),

$$f(t) = \sum_{m=-\infty}^{\infty} f(t_m) \left( \frac{\prod_{n=-\infty}^{\infty} (t - t_n)}{\prod_{n=-\infty}^{\infty} (t_m - t_n)} \right)$$

where

$$\frac{\pi'(t)}{\pi'(t_m)} \text{ is the } \bar{\Psi}_m(t) \text{ of Yen and}$$

$$\begin{aligned} t_m &= n/2W \text{ for a uniform sample time} \\ &= \tau_p \text{ for a nonuniform sample time.} \end{aligned}$$

The expression on the right-hand side is the general form for this type of sampling. The  $\bar{\Psi}_m$  can be rewritten in a form more analogous to the form of equation (2) as given by Yen. We do this in cases (a) and (b) below.

Case (a) let  $t_m = \tau_p$

Now,

$$\prod_{n=-\infty}^{\infty} (t - t_n) = \prod_{n=-\infty}^{\infty} \left( t - \frac{n}{2W} \right) \prod_{\substack{q=1 \\ q \neq p}}^N (t - \tau_q) = \frac{\prod_{n=-\infty}^{\infty} \left( t - \frac{n}{2W} \right) \prod_{q=1, q \neq p}^N (t - \tau_q)}{\prod_{q=1}^N \left( \tau_p - \frac{n_q}{2W} \right)}$$

Again,

$$\prod_{n=-\infty}^{\infty} (t_m - t_n) = \prod_{n=-\infty}^{\infty} \left( \tau_p - \frac{n}{2W} \right) \prod_{\substack{q=1 \\ q \neq p}}^N (\tau_p - \tau_q) = \frac{\prod_{n=-\infty}^{\infty} \left( \tau_p - \frac{n}{2W} \right) \prod_{q=1, q \neq p}^N (\tau_p - \tau_q)}{\prod_{q=1}^N \left( \tau_p - \frac{n_q}{2W} \right)}$$

Then,

$$\frac{\prod_{n=-\infty}^{\infty} (t - t_n)}{\prod_{n=-\infty}^{\infty} (t_m - t_n)} = \left[ \frac{\prod_{q=1, q \neq p}^N (t - \tau_q) \prod_{q=1}^N \left( \tau_q - \frac{n_q}{2W} \right)}{\prod_{q=1}^N \left( t - \frac{n_q}{2W} \right) \prod_{\substack{q=1 \\ q \neq p}}^N (\tau_p - \tau_q)} \right] \frac{\prod_{n=-\infty}^{\infty} \left( t - \frac{n}{2W} \right)}{\prod_{n=-\infty}^{\infty} \left( \tau_p - \frac{n}{2W} \right)}$$

$$= [ ] \frac{\sin 2\pi W t}{\sin 2\pi W \tau_p}.$$

Case (b) let  $t_n = n/2W \neq n_q/2W$

$$\frac{\prod_{n=-\infty}^{\infty} (t - t_n)}{\prod_{n=-\infty}^{\infty} (\frac{n}{2W} - t_n)} = \frac{\prod_{q=1}^N (t - \tau_q) \prod_{q=1}^N (\frac{n}{2W} - \frac{n_q}{2W})}{\prod_{q=1}^N (t - \frac{n_q}{2W}) \prod_{q=1}^N (\frac{n}{2W} - \tau_q)} \frac{(-1)^n \sin 2\pi W t}{\pi(2Wt - n)}.$$

As noted earlier, what we call Type III generalization has been considered by Fogel and Jagerman [8] for the forms (2) and (3) with  $K = 1$ . Most of their proof using certain boundedness conditions on  $f(z)$  involves showing that the contour integral

$$\oint_{c_n} \frac{f(\xi)}{(\xi - \zeta) \sin(\pi/h)\xi} d\xi \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

The detail is rather formidable and the motivation is carried out in a somewhat heuristic way. The devised result then comes from residue evaluation. Indeed, the appropriate restriction on  $f(z)$  for using this approach of the Cauchy integral can be found in the work of Paley and Wiener [18] and Levinson [19].

Thus, the possibility of proceeding in general to form (7) from (6) and of avoiding particular contour integration appears preferable. Aside from the possible further extensions of this approach to a multi-variable situation, as previously noted, we close this section by raising some questions and commenting on them in a way which may provide clarification and extension concerning topics either hinted at or neglected in the literature.

Question 1: Can the  $K+1$  derivatives (if they exist) taken at a  $K/2W$  spacing be chosen from any order of derivatives (as Fogel seems to suggest)? For example, can we take as the  $K+1$  derivatives  $f^{(1)}, f^{(3)}, f^{(5)}, \dots, f^{(2k+1)}$ , and, if so, what form does the interpolating formula take?

Question 2: The usual form (3) places the function and its derivative values at successive points with fixed and uniform spacing  $K/2W$ , i. e.,  $f(jK/2W), f^{(1)}(jK/2W), \dots, f^{(k)}(jK/2W)$ , at each  $j$ . Is it possible to specify these values in any other way such as

$$\begin{aligned} & \dots, f(jK/2W), f((j+1)K/2W), \dots \\ & \dots, f^{(1)}(j(K+1)/2W), f^{(1)}((j+1)(K+1)/2W), \dots \\ & \vdots \\ & \dots, f^{(k)}(j(2K-1)/2W), f^{(k)}((j+1)(2K-1)/2W), \dots \end{aligned}$$

or ( $K$  even),

$$\begin{aligned} & f^{(0)}, f^{(2)}, \dots, f^{(k)} \text{ at } jK/2W, (j+1)K/2W, \dots \\ & f^{(1)}, f^{(3)}, \dots, f^{(k-1)} \text{ at } j(K+1)/2W, (j+1)(K+1)/2W, \dots, \text{ etc. ?} \end{aligned}$$

Question 3: Considering the sampling generalizations Type I and II is it possible to combine these, that is, is it possible to introduce derivative values into nonuniform sampling?

Formally, at least, the general form of (7) appears to provide the answers to the above questions as follows:

Answer 1: If  $f^{(\ell)}$  appears then so does  $f^{(0)}, f^{(1)}, \dots, f^{(\ell-1)}$ , so that if  $K+1$  derivatives are to be specified this implies they are  $f^{(0)}, f^{(1)}, \dots, f^{(k)}$ .

Answer 2: Again if  $f^{(\ell)}(z)$  is taken at  $z=z_k$ , so also are  $f^{(0)}, f^{(1)}, \dots, f^{(\ell-1)}$ .

Answer 3: Considering the possibility of having  $m_k > 1$  and at the same time choosing the  $z_k$  to satisfy some nonuniform sample time pattern, it would appear, at least formally, that a combination of Types I and II generalizations could be developed (with the  $\phi_k(\tau)$  taking on their corresponding forms) into a more general sampling theorem than has been previously proposed.

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# A NOTE ON FANO'S INEQUALITY

T. J. Wagner

Let  $X$  and  $Y$  be two finite-valued random variables for which we try to deduce  $X$  from the occurrence of  $Y$  by any function  $g$ : range of  $Y$  to range of  $X$ . Fano's inequality<sup>†</sup> states that the probability of error  $P[g(Y) \neq X]$  is related to the conditional uncertainty  $I(X/Y)$  by

$$I(X/Y) \leq h(P[g(Y) \neq X]) + P[g(Y) \neq X] \log_2 [|X| - 1]$$

where

$$(i) \quad I(X/Y) = \sum_{\{(x, y) : P(x, y) > 0\}} -P(x, y) \log_2 \frac{P(x, y)}{P(y)}$$

$$(ii) \quad h(x) = \begin{cases} -x \log x - (1-x) \log (1-x) & 0 < x < 1 \\ 0 & x = 0, 1 \end{cases}$$

and (iii)

$|X|$  denotes the number of points  $x$  in the range of  $X$  with  $P(x) > 0$ .

We now prove a statement which can serve as the other half of Fano's inequality.

Assertion:

If a function  $g$  satisfies

$$(*) \quad P(g(y)/y) \geq P(x/y) \quad \text{for all } x, y,$$

then ,

$$P[g(Y) \neq X] \leq I(X/Y).$$

<sup>†</sup>R. M. Fano, Transmission of Information, New York: John Wiley and Sons, Inc. and the M. I. T. Press, 1961, p. 187.

Proof:

$$\begin{aligned} I(X/Y) &= \sum_{\{(x,y): P(x,y) > 0\}} -P(x,y) \log \frac{P(x,y)}{P(y)} \\ &\geq \sum_{\{(x,y): P(x,y) > 0, x \neq g(y)\}} P(x,y) \{-\log_2 P(x/y)\}. \end{aligned}$$

For each  $x, y$  such that  $x \neq g(y)$  and  $P(x, y) > 0$ ,  $-\log_2 P(x/y) \geq 1$  since  $P(x/y) \leq P(g(y)/y)$  and  $x \neq g(y) \implies P(x/y) \leq 1/2$ . Therefore,

$$\begin{aligned} \sum_{\{(x,y): P(x,y) > 0, x \neq g(y)\}} P(x,y) \{-\log_2 P(x/y)\} &\geq \sum_{\{(x,y): P(x,y) > 0, x \neq g(y)\}} P(x,y) \\ &= P[g(Y) \neq X] . \end{aligned} \quad \text{Q. E. D.}$$

Remark:

For a finite-valued random variable  $X$  and a sequence of finite-valued random variables  $Y_1, Y_2, \dots$ , let

$$\{g_n: \text{range of } (Y_1, \dots, Y_n) \text{ to range of } X\}$$

be any sequence of functions where  $g_n$ , for each  $n$ , satisfies (\*) with  $Y$  replaced by  $(Y_1, \dots, Y_n)$ . It is straightforward to show from the assertion and Fano's inequality that  $I(X/Y_1, \dots, Y_n)$  is asymptotically equal to  $P[g_n(Y_1, \dots, Y_n) \neq X]$ , that is,

$$\begin{aligned} \liminf_n -\frac{1}{n} \log_2 P[g_n(Y_1, \dots, Y_n) \neq X] \\ = \liminf_n -\frac{1}{n} \log_2 I(X/Y_1, \dots, Y_n). \end{aligned}$$